

## BFKL Dynamics in Jet Evolution

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### Abstract

We calculate  $e^+e^- \rightarrow Q\bar{Q}(k) + \text{anything}$  in a certain momentum,  $k$ , region for heavy quark-antiquark ( $Q\bar{Q}$ ) production. In our chosen region we find that the number of heavy quark pairs produced is determined by BFKL dynamics and the energy dependence of the number of pairs is given by  $\alpha_P - 1$  the hard pomeron intercept.

### 1. Introduction

The topic of this paper, BFKL dynamics [1, 2] in QCD jet evolution [3, 4] is perhaps surprising since it is generally thought that BFKL dynamics has uniquely to do with high-energy scattering and not at all to do with jet physics. Indeed, to a large extent jet evolution is an understood and well-tested subject with the dominant dynamics being given by double logarithmic perturbative terms coming from emission off the primary energetic partons. Their resummation leads to a Sudakov type of distribution. Until very recently there has been no hint of BFKL dynamics in jet physics. However, about two years ago Dasgupta and Salam [5] observed that in order to calculate certain “non-global” observables such as the probability,  $\Sigma$ , that a jet of energy  $E$  decay while emitting less than a certain amount of energy,  $E_{\text{out}}$ , into a specified angular region away from the jet, it is necessary to

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evaluate single logarithmic terms in the jet evolution. For  $\Sigma$  double logarithmic terms, soft and collinear, cancel leaving only the logarithms from soft non-collinear emissions originating from successive branching of soft gluons at comparable angles.

Stimulated by the results of Dasgupta and Salam, Banfi, Marchesini and Smye (BMS) [6] found an equation for all the single logarithmic terms contributing to  $\Sigma$  derived from the multi-soft gluon distributions given in Ref. [3]. The surprise is that the dominant part of the BMS equation is identical to the Kovchegov equation [7]. This is surprising for several reasons: (i) The Kovchegov equation is an approximate equation for dealing with small- $x$  evolution, or high-energy scattering, when unitarity limits are being reached. The BMS equation is not an approximate equation; it is an equation which exactly deals with single logarithmic terms. However, it appears to have nothing to do with unitarity limits, rather, the nonlinear aspects of the equation are related to the nature of the non-global observable  $\Sigma$ . It is due to the non linearity of successive branchings of soft gluons (or jets) at comparable angles. (ii) The linear version of the Kovchegov equation is the BFKL equation, and when scattering is far from unitarity limits the Kovchegov equation becomes the BFKL equation. There appears to be no limit in which the BMS equation reduces to the BFKL equation. The linear limit of the BMS equation is just low orders of perturbation theory.

Is it just accidental that the BMS equation is the same as an equation which naturally appears in high-energy scattering, or is it rather an indication that BFKL dynamics is also present in jet decays, waiting to be found? We believe the latter is the case, and we have found an observable in jet decays which, in certain kinematic regions, is dominated by BFKL dynamics. The observable is easiest to describe in the context of  $e^+e^-$  annihilation, although we could equally well define similar observables in deep inelastic scattering or in hadron-hadron collision events with large  $E_T$ . In the center of mass of the collision let  $E_0$  be the energy of the  $e^+e^-$  pair. The observable is the number of heavy quark-antiquark pairs produced having pair mass  $\mathcal{M}$  on the order of  $2M$ , with  $M$  the heavy quark mass, and with  $k_0$ , the energy of the pair, in the regime  $k_0/\mathcal{M}$  on the order of one. We suppose that  $E_0/k_0 \gg 1$ .

Of course most heavy quarks are not produced in the kinematic region where we are looking. The majority of heavy quarks will be collinear with one of the two primary jets and their production will be dominated by double logarithmic terms. Our choice of kinematics is motivated by the fact that

double logarithmic terms cancel out in the region we have taken. In subsequent sections, where the calculation is performed, we choose, for technical reasons, a boosted frame where the small angle approximation can be used.

In section 2, we calculate the lowest order contribution of a heavy quark-antiquark emission from a QCD dipole.

In Section 3, we calculate the evolution of the two-jet system, originally produced in the  $e^+e^-$  collision, in terms of QCD dipoles. This section makes use of the large  $N_c$  approximation, however, it may well be that similar results can be obtained without the large  $N_c$  approximation.

In section 4 we compare the two-jet evolution equation with the dipole version of QCD evolution in high-energy scattering and find them to be formally equivalent, but in terms of different variables. In jet evolution it is the angles of gluons which are frozen during subsequent emissions and hence become the natural variables, while in high-energy scattering it is transverse coordinates which are frozen and so are the natural variables.

In section 5 we give the rate of heavy quark-antiquark pair production in our chosen kinematic region, and we find that  $\alpha_P - 1$ , the hard pomeron intercept, determines the jet energy dependence of the production.

## 2. The observable and the Born approximation

The observable we consider is heavy quark-antiquark production in jet decay. The calculation will be done in the large  $N_c$  (planar) approximation where one can view jet evolution in terms of a sequence of softer and softer gluon emissions which in turn can be viewed as the production of more and more QCD dipoles. It may be that much of what we derive can also be obtained beyond the large  $N_c$  limit, but that is beyond the scope of the present paper. The simplicity of the large  $N_c$  limit is that one may view the gluon as being made of a quark part and an antiquark part.

In this section we focus on the heavy quark-antiquark pair ( $Q\bar{Q}$ ) emission from a dipole consisting of the quark part of a gluon of momentum  $p_a$  and the antiquark part of a gluon of momentum  $p_b$ . The gluons  $p_a$  and  $p_b$  have come from earlier parts of the decay of a jet, or pair of jets, and it is assumed that  $|\vec{p}_a|, |\vec{p}_b| \gg |\vec{k}| = \kappa$  with  $k_\mu$  the four-momentum and  $\sqrt{k_\mu k^\mu} = \mathcal{M}$  the mass of the  $Q\bar{Q}$  pair. We denote by  $p_1$  and  $p_2$  the momenta of the  $Q$  and  $\bar{Q}$ , respectively, with  $k_\mu = (p_1 + p_2)_\mu$ . It is important to emphasize that  $k$  is the softest momentum in our problem, softer than all other gluons emitted

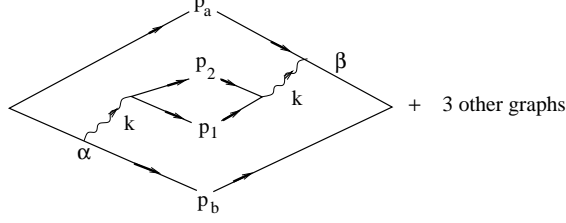


Fig. 1

in the secondary branchings, so below we are able to use the soft momentum approximation.

We now turn to the calculation of the lowest order contribution to pair production from the dipole  $(p_a, p_b)$ . We denote this contribution by  $\frac{dN_{a,b}^{(0)}}{d\mathcal{M}^2 dy}$ , where  $y = \ln \kappa/\mathcal{M}$ . One of the lowest order graphs is shown in Fig. 1 where the left-hand part of the graph denotes the amplitude and the right-hand part of the graph the complex conjugate part of the amplitude. The factors accompanying the coupling of the gluon, labelled by  $k$ , to the “quark”  $a$  or to the “antiquark”  $b$  are given by

$$g \frac{(2p_a + k)_\alpha}{(p_a + k)^2} \simeq g \frac{p_{a\alpha}}{p_a \cdot k}, \quad -g \frac{(2p_b + k)_\alpha}{(p_b + k)^2} \simeq -g \frac{p_{b\alpha}}{p_b \cdot k}, \quad (1)$$

with color factors to be put in later. In writing (1) we assume a leading soft approximation where  $|\vec{k}|/|\vec{p}_a|$  and  $|\vec{k}|/|\vec{p}_b| \ll 1$ , but we do not assume collinear emission. Indeed, we shall in the end fix  $\mathcal{M}/\kappa \equiv \mathcal{M}/|\vec{k}|$  so as to guarantee that collinear emission is unimportant.

One can write

$$\begin{aligned} \frac{dN_{ab}^{(0)}}{d\mathcal{M}^2 dy} = & g^2 \frac{C_F}{2} \int \frac{d^4 k}{(2\pi)^4 \mathcal{M}^4} \delta(k_\mu k_\mu - \mathcal{M}^2) \delta\left(y - \ln \frac{\kappa}{\mathcal{M}}\right) \\ & \left( \frac{p_{a\alpha}}{p_a \cdot k} - \frac{p_{b\alpha}}{p_b \cdot k} \right) \left( \frac{p_{a\beta}}{p_a \cdot k} - \frac{p_{b\beta}}{p_b \cdot k} \right) 2\text{Im} \pi^{\alpha\beta}(k) \end{aligned} \quad (2)$$

where

$$\begin{aligned} 2\text{Im} \pi_{\alpha\beta}(k) &= 2 (k^2 g_{\alpha\beta} - k_\alpha k_\beta) \text{Im} \pi(k^2) \\ &= \frac{g^2}{(2\pi)^2} \int \frac{d^3 p_1 d^3 p_2}{2E_1 2E_2} \delta^4(k - p_1 - p_2) \text{tr} \{ (\gamma \cdot p_1 + M) \gamma_\alpha (\gamma \cdot p_2 - M) \gamma_\beta \} \end{aligned} \quad (3)$$

with  $M$  the heavy quark mass. From (3) it is straightforward to get [8]

$$\text{Im } \pi(\mathcal{M}^2) = -\frac{\alpha_s}{3} \sqrt{\frac{\mathcal{M}^2 - 4M^2}{\mathcal{M}^2}} \frac{\mathcal{M}^2 + 2M^2}{\mathcal{M}^2}. \quad (4)$$

Because of current conservation the  $k_\alpha k_\beta$  term in  $\text{Im } \pi_{\alpha\beta}(k)$  does not contribute to (2) leaving

$$\begin{aligned} \frac{dN_{ab}^{(0)}}{d\mathcal{M}^2 dy} = & -4\pi\alpha_s C_F \int \frac{d^4 k}{(2\pi)^4 \mathcal{M}^4} \delta(k_\mu k^\mu - \mathcal{M}^2) \delta\left(y - \ln \frac{\kappa}{\mathcal{M}}\right) \\ & \cdot \frac{p_a \cdot p_b}{(p_a \cdot k)(k \cdot p_b)} 2\mathcal{M}^2 \text{Im } \pi(\mathcal{M}^2). \end{aligned} \quad (5)$$

It is now straightforward to get

$$\frac{dN_{ab}^{(0)}}{d\mathcal{M}^2 dy} = \frac{\alpha_s^2 C_F}{6\pi^3} \sqrt{\frac{\mathcal{M}^2 - 4M^2}{\mathcal{M}^2}} \frac{\mathcal{M}^2 + 2M^2}{\mathcal{M}^4} \int \frac{\theta_{ab}^2 d\Omega_k}{(\theta_{ak}^2 + \frac{\mathcal{M}^2}{\kappa^2})(\theta_{kb}^2 + \frac{\mathcal{M}^2}{\kappa^2})} \quad (6)$$

where  $\theta_{ak}$  is the angle between the vectors  $\vec{p}_a$  and  $\vec{k}$ , with similar definitions for  $\theta_{kb}$  and  $\theta_{ab}$ , and where we take  $\mathcal{M}/\kappa \ll 1$ , and we suppose  $\theta_{ab} \ll 1$  so that the small angle approximation  $\cos \theta_{ab} \simeq 1 - \frac{1}{2}\theta_{ab}^2$  along with similar approximations for  $\cos \theta_{ak}$  and  $\cos \theta_{kb}$  can be used. Finally, it is straightforward to evaluate

$$\int \frac{\theta_{ab}^2 d\Omega_k}{(\theta_{ak}^2 + \frac{\mathcal{M}^2}{\kappa^2})(\theta_{kb}^2 + \frac{\mathcal{M}^2}{\kappa^2})} = \frac{2\pi \lambda}{\sqrt{1+\lambda^2}} \ln \left[ \frac{\sqrt{1+\lambda^2} + \lambda}{\sqrt{1+\lambda^2} - \lambda} \right], \quad \lambda = \frac{\kappa \theta_{ab}}{2\mathcal{M}}. \quad (7)$$

This gives

$$\frac{dN_{ab}^{(0)}}{d\mathcal{M}^2 dy} = \frac{\alpha_s^2 C_F}{3\pi^2} \frac{\sqrt{\mathcal{M}^2 - 4M^2}}{\mathcal{M}^3} \left(1 + \frac{2M^2}{\mathcal{M}^2}\right) \frac{\lambda}{\sqrt{1+\lambda^2}} \ln \left[ \frac{\sqrt{1+\lambda^2} + \lambda}{\sqrt{1+\lambda^2} - \lambda} \right]. \quad (8)$$

We imagine choosing  $\mathcal{M}^2 - 4M^2$  on the order of  $M^2$  so that the counting of heavy quarks and heavy mesons should be the same and physically observable. We also choose  $\lambda$  on the order of one so that there are no collinear singularities in the emission of the heavy quark pair from the dipoles.

### 3. The evolution equation

The evolution (branching) [3, 9] of QCD jets has been widely studied and furnishes some of the best tests of perturbative QCD. Double logarithmic terms, having both soft and collinear singularities, dominate the behavior of most global observables, and the resummation of these perturbation series is now well understood.

The evolution we are going to review in this section is single logarithmic. Soft, but not collinear singularities, will be summed. Such terms govern the behavior of certain non-global observables such as  $E_{\text{out}}$ , the total energy emitted into a region  $C_{\text{out}}$  away from all the hard jets. The same single logarithmic terms will dominate heavy quark-antiquark production in certain regions of phase space as we shall see below. However, for the moment we are going to describe more formally the evolution that will be dominant for our heavy quark-antiquark production. We shall see that the evolution is determined by the formalism developed in by Banfi, Marchesini and Smye (BMS) [6] in their discussion of the distribution function for  $E_{\text{out}}$  mentioned above. We shall also see, perhaps surprisingly, that the evolution is formally exactly equivalent to BFKL evolution as described in the dipole formalism.

We begin by considering the process  $e^+ + e^- \rightarrow \gamma^*(q) \rightarrow \text{quark } (p_a) + \text{antiquark } (p_b)$ . We chose an unusual frame where  $\vec{q}$  is large and along the z-axis. Without loss of generality we may suppose that  $|\vec{p}_a| = |\vec{p}_b| = E$  for our zero mass quarks and that

$$\theta_{ab} \simeq \frac{\sqrt{q_\mu q^\mu}}{E} = \frac{Q}{E}$$

is small.  $Q$ , the largest physical scale, is assumed to be much greater than the heavy quark mass  $M$  introduced in the previous section.

Of course, the quark-antiquark pair  $(p_a, p_b)$  will further evolve by gluon emission. The gluon emission probabilities are known analytically [3] in the large  $N_c$  limit and in the leading soft approximation for longitudinal momenta. No assumption of a collinear approximation is necessary. In the large  $N_c$  approximation the states having  $n$  gluons in addition to the original quark-antiquark pair may be viewed as a state of  $n+1$  dipoles. For example, the state having a quark-antiquark pair and a single gluon consists of two dipoles. The first dipole is the original quark and the antiquark part of the gluon while the second dipole consists of the quark part of the gluon

and the original antiquark. Since our object is to calculate heavy quark-antiquark production, and since the heavy quark pair can only be emitted from a single dipole, the same dipole in the amplitude and in the complex conjugate amplitude, what we need to extract from the QCD evolution is only the inclusive dipole distribution caused by the gluon emissions.

Therefore, we need to compute  $n(\theta_{ab}, \theta, Y)$  the number density of dipoles of opening angle  $\theta$ , starting from the original quark-antiquark pair which has opening angle  $\theta_{ab}$ , and in a rapidity interval  $Y = \ln E/\kappa$ . (Here we take the rapidity of a quark or gluon of energy  $\omega$  to be  $y = \ln \omega/\Lambda$ .) The dipole having opening angle  $\theta$  consists of the quark part of a gluon, say  $g_1$ , and the antiquark part of a gluon,  $g_2$ . (The original quark or antiquark may also make up either the quark or antiquark part of the dipole  $\theta$ , but this is unlikely to be the case at large  $Y$ . Our formalism allows this possibility.) Let  $\omega$  be the smallest of the energies of the gluons  $g_1$  and  $g_2$ . Then in  $n(\theta_{ab}, \theta, Y)$  we require that  $\ln E/\omega \ll Y$ . The result for the distribution of heavy quark pairs is

$$\frac{dN_{ab}}{d\mathcal{M}^2 dY} = \int \frac{d\Omega_{b'}}{4\pi} n(\theta_{ab}, \theta_{a'b'}, Y) \frac{dN_{a'b'}^{(0)}}{d\mathcal{M}^2 dy}, \quad (9)$$

where  $y = \ln \kappa/\mathcal{M}$  and the details of our normalization for  $n$  will be explained later.

In order to write an equation for  $n$  we need only an explicit expression for the emission of a single gluon, of momentum  $k$ , from the original quark-antiquark pair. By a simple calculation (see for example, Eqs. 3.2-3.4 of BMS), one finds for the number of gluons  $N_g$

$$dN_g = \bar{\alpha}_s \omega d\omega \frac{d\Omega_k}{4\pi} \frac{p_a \cdot p_b}{(p_a \cdot k)(k \cdot p_b)}, \quad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}. \quad (10)$$

(In covariant gauge only the two graphs shown in Fig. 2 contribute.) Going to angular variables

$$dN_g = 2\bar{\alpha}_s \frac{d\Omega_k}{4\pi} dy \frac{\theta_{ab}^2}{\theta_{ak}^2 \theta_{kb}^2} \quad (11)$$

in the small angle approximation, and where  $y = \ln \omega/\Lambda$ .

Eq. (11) can be turned into an equation for the inclusive dipole density simply by observing that the measured dipole has as a parent dipole either

a + antiquark part of  $k$  or quark part of  $k + b$ . Thus, as illustrated in Fig. 3,

$$\frac{dn(\theta_{ab}, \theta, Y)}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\theta_{ab}^2 d\Omega_k}{\theta_{ak}^2 \theta_{kb}^2} [n(\theta_{ak}, \theta, Y) + n(\theta_{kb}, \theta, Y) - n(\theta_{ab}, \theta, Y)], \quad (12)$$

where the third term on the righthand side of (12) is the virtual contribution of the gluon  $k$ . Eq. (12) is our basic equation, and we shall come back to it when calculating an explicit asymptotic result for heavy quark-antiquark production.

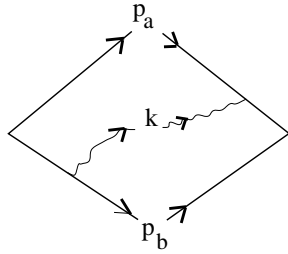


FIG. 2a

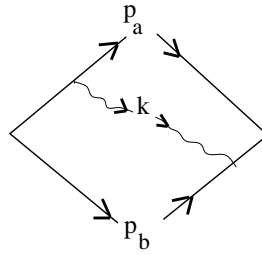


FIG. 2b

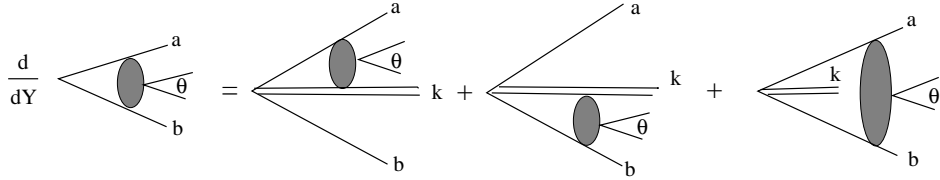


Fig. 3

In Fig. 3 the double line,  $k$ , represents the gluon,  $k$ , emitted coherently from  $a$  and  $b$ . The final term on the righthand side of Fig. 3 has the  $k$ -line being emitted and then reabsorbed so that it does not appear in the final state.

#### 4. Relationship to BFKL dynamics



Eq. (12) bears a remarkable resemblance to the BFKL equation as given in the dipole formulation [10, 11, 12]. The BFKL equation reads

$$\frac{dn(x_{01}, x, Y)}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{x_{01}^2 d^2 x_2}{x_{02}^2 x_{21}^2} [n(x_{02}, x, Y) + n(x_{21}, x, Y) - n(x_{01}, x, Y)], \quad (13)$$

for the number density of dipoles of size  $x$  to be found in a parent dipole of size  $x_{01}$  over a rapidity range  $Y$ , where the parent dipole is constructed from a quark at  $\underline{x}_0$  and an antiquark at  $\underline{x}_1$  with  $x_{01} = |\underline{x}_0 - \underline{x}_1|$ . The measured dipole has a transverse coordinate separation  $x$  between its quark and antiquark parts. In (13) we anticipate that the inclusive dipole distribution is independent of the orientations of the dipoles when  $Y$  is large. (In (12) we have similarly anticipated that  $n$  only depends on the magnitudes  $\theta_{ab}$  and  $\theta$ , but not on the orientation between the angles of the dipoles.) From (13) it is straightforward to evaluate high-energy dipole-dipole scattering whose rapidity (energy) dependence is the same as  $n$  in (13).

Except for the measure of integration  $d\Omega_k$  versus  $d^2 x_2$  equations (12) and (13) are identical. However, in the small angle limit,

$$d\Omega_k \simeq d\phi_k \theta_k d\theta_k \quad (14)$$

can be viewed as an integration measure over a flat plane with  $\theta_k$  and  $\phi_k$  the radial and polar coordinates respectively. Thus (12) and (13) are formally identical equations so long as we allow the  $\theta_k$ -integration to cover the region

$$0 < \theta_k < \infty \quad (15)$$

in (12) while still using (14). Of course when  $\theta_{ab}$  is small the important region of integration in (12) is  $\theta_k \lesssim \theta_{ab}$  so that (15) is necessary only to make (12) and (13) exactly the same.

Although (12) and (13) are formally identical the physical variables which appear in these two equations are very different. In jet evolution angles seem to be the preferred variables while in high-energy scattering transverse coordinates seem to be more natural. It is not hard to see why this is the case. In both cases the basic amplitude is the emission of a softer gluon off a higher momentum quark or gluon as illustrated in Fig. 4 for a momentum labelling of these gluons.

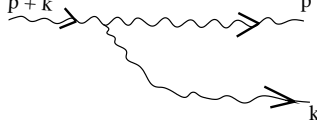


Fig. 4

In the high-energy scattering case the graph in Fig. 4 represents a gluon ( $p$ ) + gluon ( $k$ ) part of a light-cone wavefunction. The soft emission limit means  $k_+/(p+k)_+ = z \ll 1$ . The transverse momenta  $\underline{k}$  and  $\underline{p}$  are typically of the same size, and in the double logarithmic region  $|\underline{p} + \underline{k}| \ll |\underline{k}|$ . The light-cone quantization energy denominator is

$$D^{-1} = [p_- + k_- - (p+k)_-]^{-1} = \left[ \frac{\underline{p}^2}{2p_+} + \frac{\underline{k}^2}{2k_+} - \frac{(\underline{p} + \underline{k})^2}{2(p+k)_+} \right]^{-1} \simeq \left[ \frac{\underline{k}^2}{2k_+} \right]^{-1}, \quad (16)$$

and is dominated by the softest gluon,  $k$ . Thus, the time over which the gluon,  $k$ , is being emitted is

$$\tau_k \simeq \frac{2k_+}{\underline{k}^2}. \quad (17)$$

The transverse velocity of the gluon  $p$  is  $\underline{v}_p = \underline{p}/p_+$  while that of the gluon  $k$  is  $\underline{v}_k = \underline{k}/k_+$ . Over the time during which  $k$  is being emitted the changes in the transverse coordinate positions of the  $p$  and  $k$ -lines are

$$|\Delta \underline{x}_p| \simeq |\underline{v}_p| \tau_k \propto \frac{k_+}{p_+} |\underline{k}|^{-1} \quad (18)$$

and

$$|\Delta \underline{x}_k| \simeq |\underline{v}_k| \tau_k \propto |\underline{k}|^{-1} \quad (19)$$

respectively. Eq. (19) reflects the uncertainty principle while (18) says that the harder gluon has a very small change in its transverse coordinate during the emission of softer gluons. It is thus a good approximation to “freeze” the transverse coordinate of a high momentum quark or gluon in a light-cone wavefunction during the time of formation of the softer parts of the wavefunction.

On the other hand in jet evolution the energy denominator is better written as

$$D = \frac{z(1-z)}{4} (p+k)_+ [\underline{\theta}_p - \underline{\theta}_k]^2 \quad (20)$$

with

$$z = \frac{k_+}{(p+k)_+}, \quad 1-z = \frac{p_+}{(p+k)_+} \quad (21)$$

and

$$\underline{\theta}_p = \frac{\underline{p}}{p_z} = \frac{\sqrt{2}\underline{p}}{p_+}, \quad \underline{\theta}_k = \frac{\sqrt{2}\underline{k}}{k_+}. \quad (22)$$

In jet evolution  $\underline{\theta}_p$  and  $\underline{\theta}_k$  are typically of the same size, with  $|\underline{\theta}_p - \underline{\theta}_k| \ll |\underline{\theta}_p|$  in the double logarithmic limit so that typically  $|\underline{k}| \propto z|\underline{p}|$  for soft gluon emission. This means that

$$\underline{\theta}_{p+k} = \frac{\sqrt{2}(\underline{p} + \underline{k})}{(p+k)_+} \simeq \frac{\sqrt{2}\underline{p}}{p_+} = \underline{\theta}_p \quad (23)$$

for soft gluon emission. Eq. (23) indicates that the angle of a harder gluon does not change due to softer gluon emissions and so angular variables naturally appear in the jet evolution equation (12).

One final issue in comparing high-energy evolution with jet evolution is the question of the running of the QCD coupling. Equations (12) and (13) have been written for fixed coupling evolution. Running coupling effects come in very differently in (12) and (13). In (12), following BMS, we introduce a variable  $\Delta$  where

$$\Delta = \int_{E_0}^E \frac{d\omega}{\omega} \bar{\alpha}_s(\omega^2 \theta_{ab}^2), \quad \frac{\partial}{\partial \Delta} = \frac{E}{\bar{\alpha}_s} \frac{\partial}{\partial E} = \frac{1}{\bar{\alpha}_s} \frac{\partial}{\partial Y}, \quad (24)$$

and (12) becomes (replacing  $n(\theta_{ab}, \theta, Y) \rightarrow n(\theta_{ab}, \theta, \Delta)$ )

$$\frac{\partial n(\theta_{ab}, \theta, \Delta)}{\partial \Delta} = \int \frac{d\Omega_k}{2\pi} \frac{\theta_{ab}^2}{\theta_{ak}^2 \theta_{kb}^2} [n(\theta_{ak}, \theta, \Delta) + n(\theta_{kb}, \theta, \Delta) - n(\theta_{ab}, \theta, \Delta)]. \quad (25)$$

When  $\theta$  and  $\theta_{ab}$  are of similar magnitude (24) and (25) should be adequate to represent running coupling effects. We suppose, of course, that  $E > E_0$  and that  $E_0 \theta_{ab} / \Lambda_{\text{QCD}} \gg 1$ . (Note that in jet evolution the diffusion of angles away from  $\theta$  and  $\theta_{ab}$  is not a serious problem in determining the argument of the running coupling since the  $\omega$ -dependence in (24) dominates the variation of  $\bar{\alpha}$  during the evolution). Thus, in jet evolution running coupling effects are included, when  $\theta_{ab}$  and  $\theta$  are not too different, simply by the replacement

$$\bar{\alpha}_s Y \rightarrow \Delta$$

in going from the fixed coupling to the running coupling case.

Running coupling effects in high-energy BFKL applications are reasonably well understood [13, 14, 15]. For our purposes it is sufficient to observe that for  $x_{01}$  and  $x$  of comparable size one may take  $\alpha_s^{-1} \simeq \alpha_s^{-1}(x_{01}^2) \simeq -b \ln(x_{01}^2 \Lambda_{\text{QCD}}^2)$  so long as  $\alpha_s^5 Y^3 \ll 1$ . When  $\alpha_s^5 Y^3 \gg 1$  high-energy evolution and jet evolution will look quite different. We note that unitarity effects become important in the regime  $\alpha_s^5 Y^3 \ll 1$ .

## 5. The leading soft approximation result

It now only remains to combine our lowest order result for heavy quark-antiquark production from a single dipole with the solution of (12) for large values of  $Y$ . For large  $Y$  the solution of (12) is given by the standard BFKL formula [12]

$$n(\theta_{ab}, \theta, Y) = \frac{1}{\theta^2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} n_{\nu} e^{2\bar{\alpha}_s \chi(\nu) Y} \left( \frac{\theta_{ab}}{\theta} \right)^{1+2i\nu} \quad (26)$$

where

$$\chi(\nu) = \psi(1) - \frac{1}{2}\psi\left(\frac{1}{2} - i\nu\right) - \frac{1}{2}\psi\left(\frac{1}{2} + i\nu\right) \quad (27)$$

and  $n_{\nu}$  determines the normalization of  $n$  at  $Y=0$ . If we take  $n_{\nu}=4$  then

$$n(\theta_{ab}, \theta, 0) = \frac{2}{\theta^2} \delta\left(\ln \frac{\theta_{ab}}{\theta}\right) = \frac{2}{\theta} \delta(\theta - \theta_{ab}), \quad (28)$$

in which case, in a notation and normalization used in (9),

$$\int \frac{d\Omega_{b'}}{4\pi} n(\theta_{ab}, \theta_{a'b'}, Y=0) = 1. \quad (29)$$

It is straightforward to find

$$n(\theta_{ab}, \theta, Y) \underset{Y \text{ large}}{\simeq} \frac{\theta_{ab}}{\theta^3} \frac{e^{(\alpha_P-1)Y}}{\sqrt{\frac{7}{2}\pi\bar{\alpha}_s\zeta(3)Y}} e^{\frac{-\ln^2(\theta_{ab}^2/\theta^2)}{14\bar{\alpha}_s\zeta(3)Y}} \quad (30)$$

with, as usual,

$$\alpha_P - 1 = \bar{\alpha}_s 4 \ln 2. \quad (31)$$

In case of running coupling evolution, Eq. (25), one simply replaces  $\bar{\alpha}_s Y$  by  $\Delta$  in (30). However, in the running coupling case it is difficult to get large values of  $\Delta$  in realistic circumstances.

Now using (8) and (30) in (9) one arrives at the number of heavy quark pairs in the original dipole to be

$$\frac{dN_{ab}}{d\mathcal{M}^2 dY} = \frac{\alpha_s^2}{24} \left( \frac{\kappa \theta_{ab}}{\mathcal{M}} \right) \frac{e^{(\alpha_P-1)Y}}{\sqrt{\frac{7}{2}\alpha_s N_c \zeta(3) Y}} \frac{\sqrt{\mathcal{M}^2 - 4M^2}}{\mathcal{M}^3} \left( 1 + \frac{2M^2}{\mathcal{M}^2} \right), \quad (32)$$

where  $Y = \ln E/\kappa$  and we remind the reader that  $\mathcal{M}$  is the mass of the heavy quark-antiquark pair and  $\kappa = |\vec{k}|$  the pair momentum and we suppose that  $\kappa \theta_{ab}/\mathcal{M}$  is not too different from 1 in order to stay within the (angular) diffusion radius for BFKL evolution.

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